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COSMOLOGY: A Zero Density Model

of the Universe

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by

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6 refs

ABSTRACT

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Current observational data indicate, that for an arbitrary cosmological constant, the most likely relativistic universe is an oscillating one that is characterized by a negative cosmological constant and negative curvature. It is shown that a zero density universe is a valid approximation for the indicated universe, and a region of validity for the approximation as a function of the density and deceleration parameters is established. The assumption of zero density permits the derivation of closed-form relationships for the characteristics of the selected model universe and the relations connecting theory and observation. These relationships are listed for the indicated universe.

Weyl and McVittie [1] have shown that Einstein's cosmological constant, the Λ term in the field equations, is a constant of integration and therefore a necessary term in the field equations of general relativity. When the cosmological constant is accepted as a necessary term in the field equations, the solutions of the differential equations describing the expansion of the universe are in the form of elliptic integrals which means that the solutions are numerical rather than analytic. If the assumption of zero density is made, the solutions become mathematically tractable. It is shown that under certain conditions a zero density universe is a valid approximation to the universe indicated by current observational data. The relationship giving the characteristics for the most likely model universe, under the zero density approximation, are set down.

We consider isotropic, homogeneous models of the universe and take the Robertson Walker line element in the form, see reference 2,

$$ds^2 = dt^2 - \frac{R^2(t)}{C^2}\left\{d\omega^2 + s^2(\omega)(d\theta^2 + \sin^2\theta\, d\varphi^2)\right\} \tag{1}$$

where ds is measured along the line element and has the dimensions of time, t is time, C the speed of light in vacuo, $R(t)$ the scale factor that describes how the universe unfolds with time and has the dimension of length, ω, θ, φ are the dimensionless variables of the metric subspace, and $s(\omega)$ is a function that depends on the curvature of space.

The dynamics of cosmological theory are determined by Einstein's equations

$$G_{\mu\nu} - \frac{1}{2} G g_{\mu\nu} + \Lambda G_{\mu\nu} = -\kappa T_{\mu\nu} \tag{2}$$

where $G_{\mu\nu}$ is the Ricci tensor, G its spur, $T_{\mu\nu}$ the energy momentum tensor, $g_{\mu\nu}$ the metrical tensor, Λ the cosmological constant, and $\kappa = \frac{8\pi\gamma}{C^2}$. Using Dingles formula to introduce (1) into (2) we obtain the equations describing the expansion of the universe

$$\left.\begin{aligned} \frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{8\pi\gamma p}{C^2} &= -\frac{kC^2}{R^2} + \Lambda \\ \frac{\dot{R}^2}{R^2} - \frac{8\pi\gamma\rho}{3} &= -\frac{kC^2}{R^2} + \frac{\Lambda}{3} \end{aligned}\right\} \tag{3}$$

where γ is the constant of gravitation, ρ the density, p the pressure, and k a constant related to the curvature of space. The methematics having been adjusted so that k has the values +1, 0, -1 depending on the curvature of space.

The solutions of equations (3) can be obtained in the form of quadratures (3), useful information can be obtained by evaluating these equations for the present time and the subscript zero indicates the present. We define in the usual manner $H_o = \frac{\dot{R}_o}{R_o}$ and $q_o = -\frac{\ddot{R}_o}{R_o H_o^2}$. Here H_o is the Hubble parameter and has the dimensions of $(\text{secs})^{-1}$ and q_o is the acceleration parameter and is dimensionless. Setting the pressure equal to zero we find that equations (3) give for the constants Λ and k

$$\Lambda = 3H_o^2(\sigma_o - q_o) \tag{4}$$

$$\frac{kC^2}{R_o^2} = H_o^2(3\sigma_o - q_o - 1) \tag{5}$$

where σ_o is the density parameter and $\sigma_o = \frac{4\pi\gamma\rho_o}{3H_o^2}$. Present observational data establishes H_o, q_o, and ρ_o as

$$0.5 \leq q_o \leq 2.5$$
$$2.43 \times 10^{-18} \leq H_o \leq 4.86 \leq 10^{-18}$$
$$3 \times 10^{-31} \leq \rho_o \leq 3 \times 10^{-30}$$

Insertion of these data into equations (4) and (5) indicate that for the range of variation shown above, that the universe is an oscillating

one and is characterized by negative values of Λ and k. The value of k shows that space is hyperbolic and the $s(\omega)$ in equation (1) is $\sinh \omega$.

We wish to use equations (1) and (3) to determine (a) distances in the universe, and (b) the form of $R(t)$. Two basic integrals are encountered, the first

$$\omega = \frac{C}{R_0 H_0} \int_0^\delta \frac{d\delta}{\left\{1 + 2(q_0 + 1)\delta + (3\sigma_0 + q_0 + 1)\delta^2 + 2\sigma_0\delta^3\right\}^{1/2}} \tag{6}$$

along the null geodesic, $ds = 0$, of light travel when integrating the metric and the second

$$\tau = t_0 - t = T_0 \int_0^\delta \frac{d\delta}{(1 + \delta)\left[1 + 2(q_0 + 1)\delta + (3\sigma_0 + q_0 + 1)\delta^2 + 2\sigma_0\delta^3\right]^{1/2}} \tag{7}$$

when integrating the second of equations (3) for time.

In equations (6) and (7) δ is the red shift and is dimensionless, t is any time except t_0, and $T_0 = H_0^{-1}$ and has the dimensions of seconds. In both of these cases it is apparent that terms involving σ_0 are the ones causing the integrals to be elliptic. On the basis of present observational data $\sigma_0 \approx 0.01$ which is small compared to one. If the σ_0 terms can be neglected in equations (6) and (7), these equations can be readily integrated in terms of simple functions. Figures 1 and 2 compare the $\frac{\omega R_0 H_0}{C}$ and $\frac{\tau}{T_0}$ for $q_0 = 0.5$ and $\sigma_0 = 0$ and 0.08.

($\rho_o = 3 \times 10^{-30}$ g/cc). H_o was taken as 100 km/sec/10^6 parsec. In these figures, because of the small differences involved, only those parts of the curve where differences occur are shown. As can be seen, the greatest error occurs at $\delta = 1$ and in about 1 percent, the errors when $q_o = 2.5$ (not shown) are must less. These results indicate that a zero density universe is an excellent approximation to the actual universe when $\sigma_o \leq 0.08$, $k = -1$, and $\Lambda < 0$. Further calculations showed that if σ_o meets the condition

$$\sigma_o \leq 0.075\, q_o(q_o - 1) + 0.1 \tag{8}$$

that the approximation is still very good and for the range of q_o and δ between zero and one the error is less than 2 percent. Equation (8) establishes, in view of the uncertainty of observational data, a conservative region of validity for the zero density approximation. As noted, precise values for H_o and ρ_o have not yet been established. The use of σ_o establishes a strong boundary since variations in both of these parameters are taken into account.

Using the $\sigma_o = 0$ assumption, equations (6) and (7) can be integrated. The results are

$$\omega = \cosh^{-1}\sqrt{\frac{q_o + 1}{q_o}}\,(1 + \delta) - \cosh^{-1}\sqrt{\frac{q_o + 1}{q_o}} \tag{9}$$

and

$$\tau = \frac{T_o}{\sqrt{q_o}}\left\{\cos^{-1}\sqrt{\frac{q_o}{q_o + 1}}\left(\frac{1}{1 + \delta}\right) - \cos^{-1}\sqrt{\frac{q_o}{q_o + 1}}\right\} \tag{10}$$

With these two integrals evaluated it is a matter of routine to write down the relationship for the zero density model universe.

Using equations (4) and (5), we find that the most likely universe is an oscillating one that is characterized by $\Lambda < 0$ and $k = -1$. We assume that $\sigma_o = 0$ and the constants Λ and k are given by

$$\Lambda = -3H_o^2 q_o \tag{11}$$

$$\frac{kC^2}{R_o^2} = -H_o^2(q_o + 1) \tag{12}$$

depending on the sign and magnitude of q_o several zero density model universes are possible. However, when observational data are substituted, equations (11) and (12) indicate the same universe as equations (4) and (5). The rest of this paper will discuss this specific universe where $\Lambda < 0$, $k = -1$.

Using equation (10) $R(t)$ is given by

$$R = \frac{CT_o}{\sqrt{q_o + 1}}\left\{\cos\frac{\sqrt{q_o}(t_o - t)}{T_o} - \frac{1}{\sqrt{q_o}}\sin\frac{\sqrt{q_o}(t_o - t)}{T_o}\right\} \tag{13}$$

the time since the beginning of expansion, t_o, is

$$t_o = \frac{T_o}{\sqrt{q_o}}\left\{\frac{\pi}{2} - \cos^{-1}\sqrt{\frac{q_o}{q_o + 1}}\right\} \tag{14}$$

and the time interval $t_o - t$ is given by

$$t_o - t = \frac{T_o}{q_o}\left\{\cos^{-1}\sqrt{\frac{q_o}{q_o + 1}} - \cos^{-1}\sqrt{\frac{q_o}{q_o + 1}\left(\frac{1}{1 + \delta}\right)}\right\} \tag{15}$$

There are two distances of interest in cosmology, the luminosity distance $D_l = R_o s(\omega)(1 + \delta)$ and derives from the definition of apparent luminosity and the cosmic distance $D_C = R_o s(\omega)$ which connects events for constant cosmis time, in this case the epoch of observation, see reference 4. Equation (9) is used in connection with $s(\omega) = \sinh \omega$ to obtain these relations. Thus

$$D_l = \frac{CT_o(1 + \delta)}{q_o}\left\{\sqrt{(1 + q_o)(1 + \delta)^2 - q_o} - (1 + \delta)\right\} \tag{16}$$

and

$$D_C = \frac{CT_o}{q_o}\left\{\sqrt{(1 + q_o)(1 + \delta)^2 - q_o} - (1 + \delta)\right\} \tag{17}$$

With these distances specified we can write down three relations connecting observation and theory. The most important, the red shift magnitude relation is

$$m_{bol} = 5 \log\left\{\frac{CT_o(1 + \delta)}{q_o}\left[\sqrt{(1 + q_o)(1 + \delta)^2 - q_o} - (1 + \delta)\right]\right\} + M_{bol} - 5 \tag{18}$$

where m_{bol} is the apparent bolometric magnitude and M_{bol} is the absolute bolometric magnitude at 10 parsecs. The number count relation, the number of galaxies brighter than a given magnitude is

$$N(m_{bol}) = \frac{2\pi n C^3}{QH_o^3(q_o + 1)^{3/2}}\left\{\frac{p - 1}{2}\,\Omega^2\sqrt{\Omega^{-2} + \left(\frac{p - 1}{2}\right)^2} - \sinh^{-1}\left[\left(\frac{p - 1}{2}\right)\Omega\right]\right\} \tag{19}$$

where

$$p = 1 + \frac{2H_o 10^{0.2(m_{bol} - M_{bol})+1}}{c} \tag{19-1}$$

$$\Omega = \frac{q_o + 1}{\frac{p}{2}\left(1 + [p - 1]\sqrt{\frac{q_o}{p^2} + \left(\frac{1}{p-1}\right)^2}\right)} \tag{19-2}$$

and Q is the number of square degrees in the celestial spheres.

The last of the relationships connecting theory and observation is angular diameter, ϵ, redshift relation which, using McVittie's approximation [2], for the linear diameter, is

$$\epsilon = \frac{H_o q_o (1 + \delta) \cdot 10^{-0.2 M_{o_{bol}} + 1}}{c\left(\sqrt{(q_o + 1)(1 + \delta)^2 - q_o} - (1 + \delta)\right)} \tag{20}$$

where $M_{o_{bol}}$ is the absolute bolometric magnitude at 10 parsec of the equivalent local source.

It should be pointed out that equations (18) and (19) cannot be used directly in reducing observational data because they contain M_{bol}, the absolute bolometric magnitude of the source at the epoch of emission. This must be replaced by $M_{o_{bol}}$, the absolute bolometric magnitude of an equivalent local source, and a correction to account for the red shift and process differences between the actual and local sources. McVittie [2], Robertson [3], and Davidson [5] have discussed and present adequate methods for this correction.

When the cosmical constant is accepted as a necessary term in the field equations of general relativity, it appears in the equations describing the expansion of the universe. When these equations are evaluated for the present epoch current observational data indicate an oscillating universe characterized by a negative cosmical constant and negative curvature for space. Space is hyperbolic. The characteristics of this universe must be obtained by evaluating elliptic integrals. It has been shown that there is a zero density universe with the same characteristics as the indicated universe. The approximation is valid for $\sigma_0 \leq 0.075\ q_0(q_0 - 1) + 0.1$ and introduces an error of less than 2 percent in equations (6) and (7). The characteristics of the approximating zero density universe are readily obtained by integrating simple functions. The characteristics of the zero density universe with $\Lambda < 0$ and $k = -1$ were given. [6]

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Figure 1. Comparison of equation 6 for $\sigma_o = 0.08$ and $\sigma_o = 0$, $q_o = 0.5$, $H_o = 100$ km/sec/10^6 parsec.

Figure 2. Comparison of equation 7 for $\sigma_o = 0.08$ and $\sigma_o = 0$, $q_o = 0.5$, $H_o = 100$ km/sec/10^6 parsec.

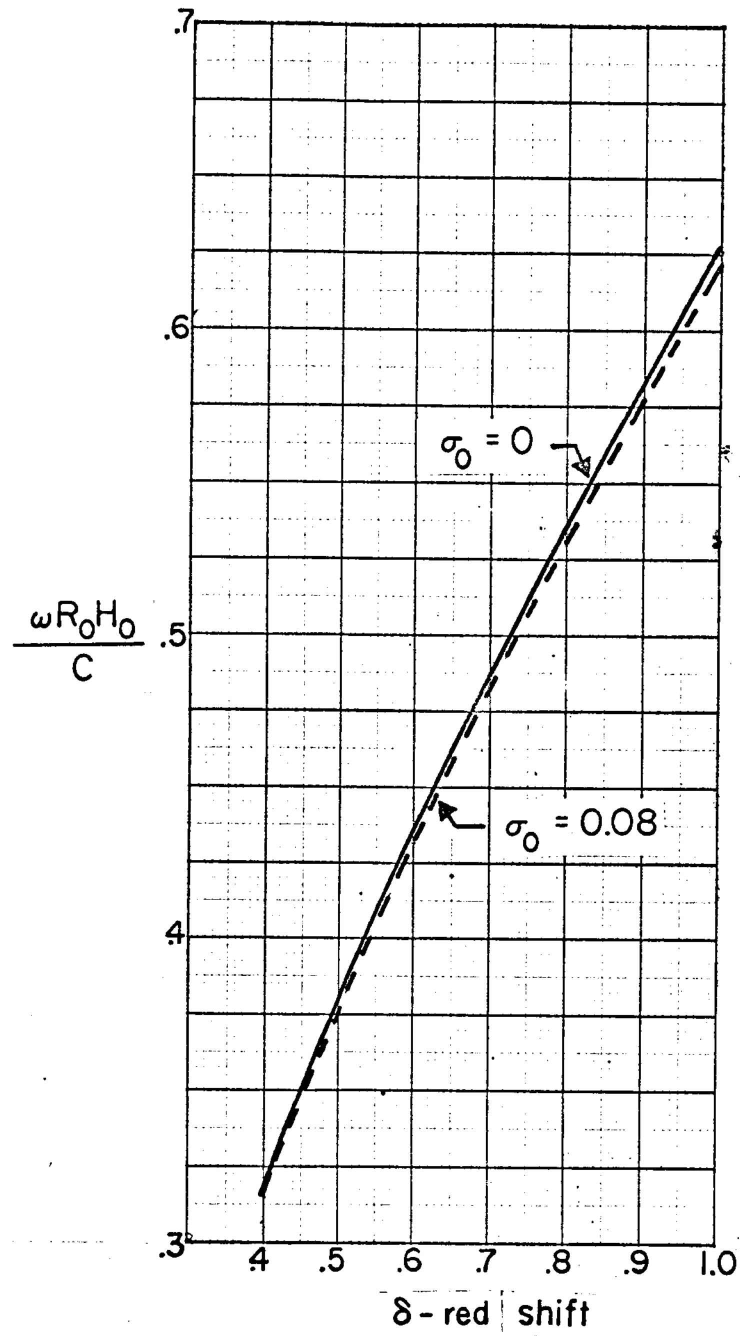
.7
.6
.5
.4
.3
$\frac{\omega R_0 H_0}{C}$
$\sigma_0 = 0$
$\sigma_0 = 0.08$
.4
.5
.6
.7
.8
.9
1.0
δ - red shift

Figure 1

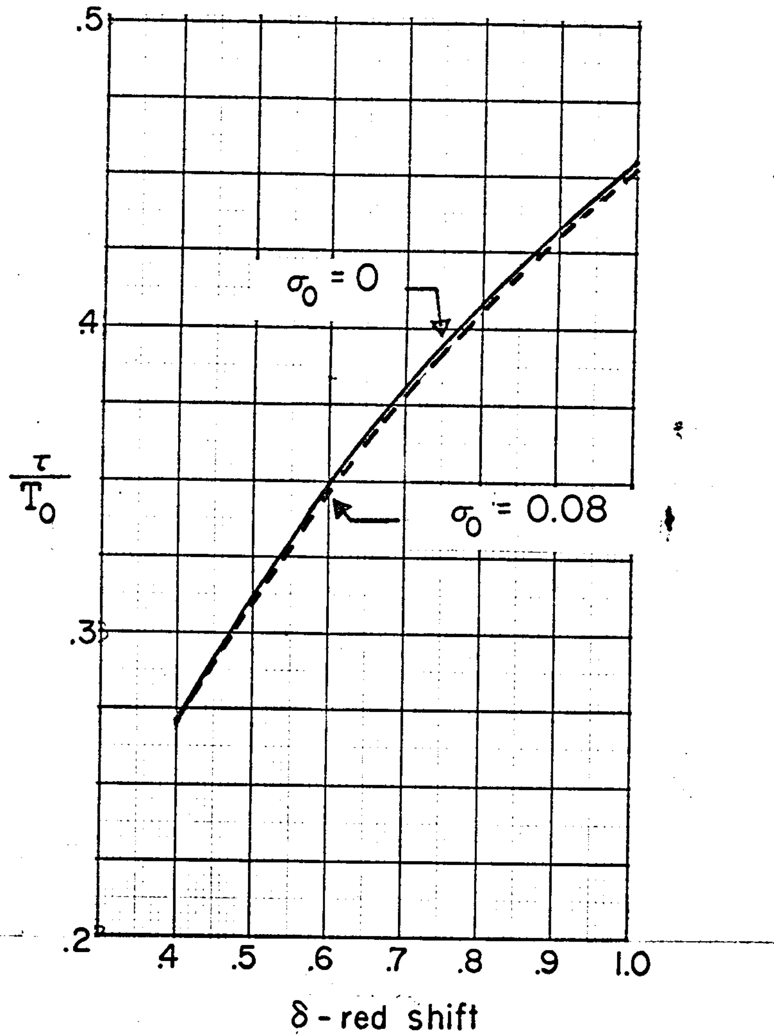

Figure 2